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Adiabatic charge transport, the eta invariant, and Hall conductance for spinors

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Abstract. We calculate the ground-state degeneracy for Schrödinger–Pauli electrons on various non-compact Riemann surfaces, as a function of a parameter representing a flux. We use the results to calculate the charge transport, and the (appropriately defined) Hall conductance, for the (degenerate) lowest Landau level of Schrödinger–Pauli electrons on these surfaces. We connect the charge transport with the Atiyah–Patodi–Singer η-invariant for (compact) manifolds with boundaries.

1. Introduction

We calculate the charge transport of the lowest Landau level for Schrödinger–Pauli electrons on various surfaces. We do this by following the flow of states caused by an adiabatically changing flux tube. The flux tube may pierce the surface or may thread it without piercing. We demonstrate how to use knowledge about the ground-state degeneracy to get information on the charge-transport properties.

The degeneracy of the ground state for spinors enables us to perform the necessary calculations, using a slight modification of the Atiyah–Patodi–Singer (APS) index theorem for manifolds with a boundary. In particular, we connect the charge transport with the APS η-invariant [APS].

The paper is organized as follows. In section 2 we briefly sketch the calculation of the Hall conductance for the Euclidian plane and the infinite cylinder, in the presence of a constant, perpendicular magnetic field B. We explain the connection between the adiabatic transport and the Hall conductance. The wish to generalize these results leads us to consider spinors. In section 3 we derive the Schrödinger–Pauli equation for non-relativistic spinors on a surface. We show that the ground-state degeneracy for a Schrödinger–Pauli operator equals the number of zero modes of the corresponding Dirac operator. In section 4 we calculate the charge transport for spinors on the plane (or any surface homeomorphic to it), under the action of a (compactly supported) magnetic field. For this we derive a generalization of a result by Aharonov and Casher [AC] which gives the ground-state degeneracy for electrons in a magnetic field on the plane, in the absence of magnetic flux tubes. In section 5 we calculate the charge transport for Schrödinger–Pauli electrons on a *general* surface with cylindrical 'ends', by using the APS index theorem for manifolds with boundaries [APS].

Throughout this paper, we adopt the unit system $\hbar = c = 2m_e = 1$, where $\hbar$ is Planck's constant, c the velocity of light, and m_e the electron's mass. In these units, ϕ_0 (one flux quantum) equals 2π. We also absorb the electron charge in the definition of the magnetic vector potential $\boldsymbol{A}$ and the magnetic field $\boldsymbol{B}$. (Boldface letters indicate forms—$\boldsymbol{A}$ is a 1-form, and $\boldsymbol{B} = \mathrm{d}\boldsymbol{A}$ is a two-form).

2. Adiabatic transport on the plane and the cylinder—a constant magnetic field

For the special cases of the plane and the cylinder, with a *constant* magnetic field B perpendicular to them, we can calculate *explicitly* the adiabatic charge transport.

2.1. Adiabatic transport on the plane (for a Schrödinger particle)

Consider the infinite x–y plane. We pierce it by a flux tube through the origin, carrying a flux ϕ. The flow of states during an adiabatic increase of the flux from zero to 2π was originally calculated in [Lau]. A detailed derivation is given in [AP]. Here, we sketch the results.

(1) For $\phi = 0$, the eigenfunctions of the Schrödinger operator can be labelled by two indices, n and m, where $n \geqslant 0$ indicates the energy, $E_{nm} = (2n+1)B$, and m is the angular momentum in the $\hat{z}$ direction, $-n \leqslant m$.

(2) While we increase the piercing flux, the states are changed according to

$$m \to m - n_\phi \qquad n \to \begin{cases} n & \text{if } m > 0 \\ n + n_\phi & \text{if } m \leqslant 0 \end{cases} \tag{1}$$

where $n_\phi \equiv \phi/2\pi$.

Hence, while the flux is increased adiabatically, the eigenstates of the system change. At the end of the process, when $n_\phi = 1$, they are mapped to themselves, as they should be. But, since this map is non-trivial, and since decreasing the quantum number m decreases the expectation value of r, there is a flow of states toward the origin: each Landau level gains a state from spatial infinity. (In addition, there is a spectral flow; see (1) and [AP].)

2.2. Adiabatic transport on the cylinder (for a Schrödinger particle)

Consider an infinite cylinder of radius 1, threaded by a flux ϕ . We define a coordinate system on it, (x, y), with $-\infty < y < \infty$, $0 \leqslant x < 2\pi$. We choose $A = By\,\mathrm{d}x$.

The time-independent Schrödinger equation is $E\psi = \left[(-\mathrm{i}\partial_x - By)^2 - \partial_y{}^2\right]\psi$. The flux enters through the boundary condition

$$\psi(x = 2\pi) = \mathrm{e}^{-\mathrm{i}\phi}\psi(x = 0)\,. \tag{2}$$

The L^2 solutions to this equation are

$$\psi_{n,k}(x, y) = F_{n,k}(y)\,\mathrm{e}^{ikx} \qquad F_{n,k}(y) = \mathrm{e}^{((By^2)/2-(ky/2\pi))}\frac{\mathrm{d}^n}{\mathrm{d}y^n}\,\mathrm{e}^{-(By^2-2(ky/2\pi))} \tag{3}$$

(where $k = l - n_\phi$, and l is an arbitrary integer), with energies $E_{n,k} = B(2n+1)$.

When the flux is increased adiabatically, the eigenstates of the system change. At the end of the process, when $n_\phi = 1$, they are mapped to themselves by the transformation: $k \to k-1$. By (3), this is equivalent to the transformation: $y \to y + (1/2\pi B)$.

Notice that, unlike the case of the plane, *no state changes its energy* during the process; there is no flow in energy space. But there is a spatial flow: each state, in all the Landau levels, moves a distance of $1/2\pi B$ in the $\hat{y}$ direction. The net effect is that one charge per each level is transferred from $y = -\infty$ to $y = +\infty$.

Here, it is natural (see the next subsection) to define the *Hall conductance* as the net number of states transferred. We get (again) that it equals 1 for each Landau level, in agreement with the result obtained by the usual definition: $\sigma_{\mathrm{Hall}} \equiv I_{\mathrm{Hall}}/V_{\mathrm{Hall}}$ (where we define the 'adiabatic voltage' to be $V = -\mathrm{d}\phi/\mathrm{d}t$.)

2.3. The adiabatic transport and the hall conductance

We can view the adiabatic charge transport as a spectral property of the Hamiltonian. For example, we may ask a question such as: given a family of eigenvalue problems, parametrized by a number ϕ, with the property $\mathrm{Spec}(H, \phi) = \mathrm{Spec}(H, \phi \bmod 2\pi)$, what is the (spectral and spatial) flow of states due to adiabatic changes in the fluxes (the spatial flow of states being related to the adiabatic transport)?

But, we can also view this same property differently: the 'parameter' ϕ represents a magnetic flux tube. Classically, when a flux ϕ is varied in time, an *electromotive force* (EMF) around it is formed. We may view the adiabatic change of a parameter ϕ as a source of a tiny EMF. From this viewpoint, the adiabatic charge transport is related to an (averaged) conductance, the number of states transported being the averaged conductance in units of $e^2/2\pi$.

In the examples we consider in this paper, we find that the direction of the charge transport is *perpendicular* to this 'EMF'. Hence, the associated conductance is a '*Hall conductance*'.

3. The Schrödinger–Pauli equation

We derive the non-relativistic approximation to the Dirac equation on two-dimensional surfaces, the Schrödinger–Pauli equation.

We choose the metric tensor to be conformal: $g_{\mu\nu} = e^{2\sigma}\delta_{\mu\nu}$, where σ is a function of the coordinates.

We write the time-dependent Dirac equation in the form (see, for example, [GSW])

$$(\not{D} + \beta m)\psi = \mathrm{i}\frac{\partial \psi}{\partial t} \tag{4}$$

where

$$\not{D} = \begin{pmatrix} 0 & K^\dagger \\ K & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5}$$

and

$$K = -\mathrm{i}\,e^{-\mathrm{i}\sigma}(2\partial_{\bar{z}} + \partial_{\bar{z}}\sigma - 2\mathrm{i}a) \qquad K^\dagger = -\mathrm{i}\,e^{-\mathrm{i}\sigma}(2\partial_z + \partial_z\sigma - 2\mathrm{i}\bar{a})\,. \tag{6}$$

We use the notations $z \equiv x_1 + \mathrm{i}x_2$, $\bar{z} \equiv x_1 - \mathrm{i}x_2$, $a \equiv \frac{1}{2}(A_1 + \mathrm{i}A_2)$, $\bar{a} \equiv \frac{1}{2}(A_1 - \mathrm{i}A_2)$. (There is a freedom in the definition of the Dirac operator [GSW]. We made a choice such that the Dirac operator is off-diagonal. For details, see [Pnu].)

Applying a routine procedure, (see, for example, [BD]), we get the non-relativistic approximation to the Dirac equation, the Schrödinger–Pauli equation:

$$\frac{\not{D}^2}{2m_e}\varphi_{\mathrm{nr}} = -\mathrm{i}\frac{\partial \varphi_{\mathrm{nr}}}{\partial t} \tag{7}$$

where φ_{nr} is a two-component spinor wavefunction, representing a non-relativistic Pauli particle.

This shows the relevance of the index of the *Dirac* operator for calculating the ground-state degeneracy of the corresponding *Schrödinger–Pauli* operator, $\not{D}^2$: the Dirac index is the difference between the dimensions of $\mathrm{Ker}K$ and $\mathrm{Ker}K^\dagger$, while the ground-state degeneracy of the Schrödinger–Pauli operator is their sum. We want to calculate the charge transport (and the Hall conductance) for the ground state of Schrödinger–Pauli operators. We begin with a simple example.

4. Adiabatic transport of spinors on the plane—a general magnetic field

We calculate the Hall conductance of the lowest Landau level on the plane, with a non-constant, compactly supported magnetic field. Our strategy is similar to the one applied in section 2: we calculate the ground-state degeneracy as a function of a piercing flux tube, and find the flow of states. We generalize this result to any non-compact, simply connected surface.

4.1. The ground-state degeneracy for a Schrödinger–Pauli operator on the plane

When speaking of 'the lowest Landau level', we refer to L^2 solutions of the Schrödinger–Pauli equation, of zero energy. The dimension of the corresponding function space is $\mathrm{Dim}\,\mathrm{Ker}_{L^2}\ K + \mathrm{Dim}\,\mathrm{Ker}_{L^2}\ K^\dagger$.

We calculate the ground-state degeneracy on the plane, with a non-constant magnetic field of compact support.

Because, on the plane $\sigma = 0$ $(g_{\mu\nu} = \delta_{\mu\nu})$,

$$-\mathrm{i}K = 2(\partial_{\bar z} - \mathrm{i}a) \qquad -\mathrm{i}K^\dagger = 2(\partial_z - \mathrm{i}\bar a) \tag{8}$$

where we use the notation

$$z = x + \mathrm{i}y = r\,\mathrm{e}^{\mathrm{i}\theta} \qquad a = \tfrac{1}{2}(A_x + \mathrm{i}A_y)\,. \tag{9}$$

We define φ:

$$-\mathrm{i}a = \partial_{\bar z}\varphi \qquad \mathrm{i}\bar a = \partial_z\varphi \tag{10}$$

$(A_x = -\partial_y\varphi,\ A_x = \partial_y\varphi$ and $B = 4\partial_{z\bar z}\varphi)$. Hence we may write

$$\mathrm{i}K = 2(\mathrm{e}^{-\varphi}\partial_{\bar z}\,\mathrm{e}^{\varphi}) \qquad \mathrm{i}K^\dagger = 2(\mathrm{e}^{\varphi}\partial_z\,\mathrm{e}^{-\varphi})\,. \tag{11}$$

From this it is easy to deduce

$$\mathrm{i}K\psi_1 = 0 \to \psi_1 = \mathrm{e}^{-\varphi}f(z) \qquad \mathrm{i}K^\dagger\psi_2 = 0 \to \psi_2 = \mathrm{e}^{\varphi}g(\bar z) \tag{12}$$

where $f(z),\ g(\bar z)$ are (arbitrary) functions of z and $\bar z$, respectively.

We want to find the dimension of the space of *square-integrable* solutions of these equations. First we do this for a magnetic field with *no flux tubes* (we call such a field *regular*). Transforming to a polar coordinate system, we find

$$A_\theta = +r\partial_r\varphi \qquad A_r = -\frac{1}{r}\partial_\theta\phi\,. \tag{13}$$

Asymptotically, for $r \to \infty$, we can choose $\varphi(r,\theta) \to \varphi(r)$, independent of θ (remember that B has a compact support!). Hence, if we take the closed path of integration as a circle of radius $r \to \infty$, centred at the origin, then:

$$\oint A_\theta\,\mathrm{d}\theta = 2\pi r\partial_r\phi = \iint B\,\mathrm{d(area)} \equiv \phi_B \tag{14}$$

where the surface integration is over the area enclosed by the path of the line integration, and ϕ_B denotes the total magnetic flux through the surface. We demand $\phi_B < \infty$.

Hence, *asymptotically*, for $r \to \infty$,

$$\varphi \simeq \frac{\phi_B}{2\pi}\ln r \equiv n_B \ln r \tag{15}$$

where n_B denotes the number of magnetic flux quanta through the surface.

Combining this with (12), we get the asymptotic form of the solutions:

$$\psi_1 \to r^{-n_B}f(z) \qquad \psi_2 \to r^{n_B}g(\bar z)\,. \tag{16}$$

Without loss of generality let us assume for the moment that $n_B > 0$. If we now impose the L^2 condition, we find that there is no square-integrable solution with $\psi_2 \neq 0$ ($g(\bar{z})$ must be a polynomial of a non-negative integer order, for single-valuedness and regularity at the origin). We can have square-integrable solutions with $\psi_1 \neq 0$, with $f(z)$ being a polynomial of order smaller than $d_{\max} = n_B - 1$. The number of such independent polynomials is $\lfloor n_B \rfloor$, which is also the ground-state degeneracy. ($\lfloor x \rfloor$ denotes the largest integer strictly smaller than x.) For negative n_B the ground-state degeneracy is $\lfloor -n_B \rfloor$.

Up to now we have reconstructed the Aharonov–Casher [AC] result: the ground-state degeneracy for the Schrödinger–Pauli operator on the plane is the largest integer strictly smaller than the absolute value of the net number of flux quanta through the plane. All these zero-energy states have the same direction of spin, depending only on the sign of n_B. This seems like a surprising result: take two large, separate areas, where there are $+n_1$ magnetic flux quanta through one, and $-n_2$ through the other (n_1, n_2 being positive), *a priori* we could assume there would be $n_1 + n_2$ zero-energy spinors, pointing in opposite directions in each area. But, as we have seen, this is not so.

Now we want to calculate the Hall conductance, or the spatial flow of states, as a function of a piercing flux at the 'origin', ϕ. For this we generalize the Aharonov–Casher result, and calculate the ground-state degeneracy as a function of the *flux*.

Again, we use $-ia = \partial_{\bar{z}}\varphi$. For $r \to 0$, we can choose φ to be independent of r. Hence, $A_\theta = r\partial_r\varphi$, and $A_r = 0$. Because $\lim_{r\to 0} \oint A_\theta \, \mathrm{d}\theta = \phi$, we find that

$$\lim_{r\to 0} \varphi = \frac{\phi}{2\pi} \ln r \equiv n_\phi \ln r \,. \tag{17}$$

Remember, we are interested in square-integrable functions of the form: $\psi_1 = \mathrm{e}^{-\varphi} f(z)$, $\psi_2 = \mathrm{e}^{\varphi} g(\bar{z})$.

Because of the flux, the wavefunctions are no longer regular at the origin, so we do not demand that $f(z)$ and $g(\bar{z})$ are of positive order. But we have to check the square integrability both at infinity and at the origin.

Assume that $n_B > 0$. Then, the square-integrability condition at infinity yields that there are no L^2 solutions with $\psi_2 \neq 0$. To have a solution with $\psi_1 \neq 0$, $f(z)$ must be a polynomial of degree less than the total magnetic flux through the surface, $n_B + n_\phi$ (we use n_B to denote only the contribution of the 'regular' part of the magnetic field, *not* including the flux tube, to the total magnetic flux through the surface).

Because very close to the origin, $\psi_1 = \mathrm{e}^{-\varphi} f(z) \to r^{-n_\phi} f(z)$, the square integrability implies that $f(z)$ is a polynomial of a degree greater than $n_\phi - 1$.

These two conditions together gives us the square-integrability condition: $f(z) = \sum_{n=n_{\min}}^{n_{\max}} a_n z^n$ with $n_{\min} > n_\phi - 1$, and $n_{\max} < n_B + n_\phi - 1$.

The dimension of the space spanned by these polynomials is $\lfloor n_B + n_\phi \rfloor - [n_\phi]$ ($[x]$ denotes the integer part of x; it is the largest integer smaller than or equal to x).

This gives us the generalization we need: in the presence of a flux tube, we also get only one spin state for the L^2 zero-energy solutions, and the ground-state degeneracy (GSD) is

$$\mathrm{GSD} = \lfloor n_B + n_\phi \rfloor - [n_\phi] \,. \tag{18}$$

We see that the ground-state degeneracy is *periodic* in ϕ, as it should be. (For negative n_B we get $\mathrm{GSD} = \lfloor -n_B + n_\phi \rfloor - [n_\phi]$, and the ground state is composed of states only with the *opposite* spin direction, $\psi = (0, \psi_2)^t$.)

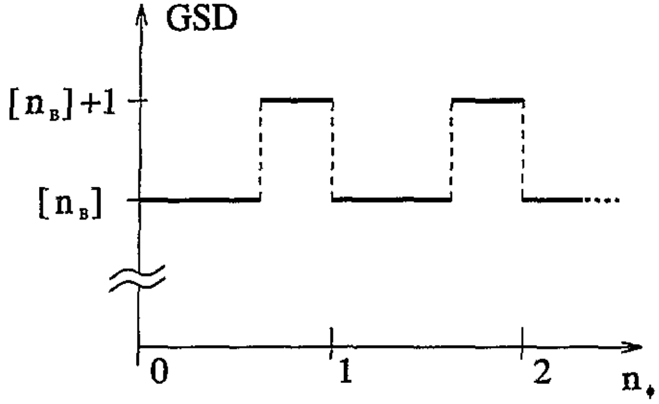

Figure 1. The ground-state degeneracy of the Pauli operator on the plane.

4.2. The Hall conductance

To calculate the Hall conductance, let us follow the change in the ground-state degeneracy as we change, adiabatically, the piercing flux from zero to one flux quanta. As long as $n_B + n_\phi$ does not cross an integer value the degeneracy is the same as for zero flux. When it crosses an integer the degeneracy increases by one, and decreases back when the flux equals one flux quanta. When we continue to increase the flux we get the same behaviour, periodically, as demonstrated in figure 1.

What happens during this process? When $n_B + n_\phi$ assumes an integer value, one state 'becomes' an L^2 state. When n_ϕ is an integer, one L^2 state increases its energy and no longer belongs to the ground state. The spatial flow is *towards the origin*.

4.3. Generalization to a curved surface

We can further generalize the Aharonov–Casher result, and calculate the ground-state degeneracy and Hall conductance for any smooth surface, homeomorphic to the plane, with vanishing Gaussian curvature at infinity, with one piercing flux through it, and a compactly supported B.

We choose a conformal metric tensor on the surface: $ds^2 = e^{2\sigma}(dx^2 + dy^2)$. The vector potential 1-form is $A = A_x\,dx + A_y\,dy$. The magnetic field 2-form is

$$B = dA = (\partial_x A_y - \partial_y A_x)\,dx \wedge dy = e^{-2\sigma}(\partial_x A_y - \partial_y A_x)\,d(\text{area}) \equiv B\,d(\text{area}). \tag{19}$$

For the choice $a = i\partial_{\bar z}\varphi$ ($B = 4e^{-2\sigma}\partial_{z\bar z}\varphi$, and $A_\theta = r\partial_r\varphi$, $A_r = -(1/r)\partial_\theta\varphi$), we have

$$\begin{aligned} i e^{\sigma} K &= 2\left(\partial_{\bar z} + \partial_{\bar z}(\tfrac{1}{2}\sigma + \varphi)\right) = e^{-((\sigma/2)+\varphi)}\partial_{\bar z}\, e^{((\sigma/2)+\varphi)} \\ -i e^{\sigma} K^{\dagger} &= 2\left(\partial_z + \partial_z(\tfrac{1}{2}12\sigma - \varphi)\right) = e^{(-(\sigma/2)+\varphi)}\partial_z\, e^{((\sigma/2)-\varphi)}. \end{aligned} \tag{20}$$

Hence

$$K\psi_1 = 0 \to \psi_1 = e^{-((\sigma/2)+\varphi)} f(z) \qquad K^{\dagger}\psi_2 = 0 \to \psi_2 = e^{-((\sigma/2)-\varphi)} g(\bar z). \tag{21}$$

We want to find the ground-state degeneracy as a function of the total magnetic and geometric fluxes through the surface (by 'geometric flux' we mean $\iint k\,d(\text{area})$, where k denotes the Gaussian curvature of the surface: $k = -4e^{-2\sigma}\partial_{z\bar z}\sigma$). To do so, we write the integrated curvature as a function of σ: we define

$$c \equiv \tfrac{1}{2}(C_x + iC_y) = -i\partial_{\bar z}\sigma \tag{22}$$

$C_\theta = -r\partial_r\sigma$, $C_r = (1/r)\partial_\theta\sigma$. Using $k = -\mathrm{e}^{-2\sigma}(\partial_{xx} + \partial_{yy})\sigma$, we find

$$k = \mathrm{e}^{-2\sigma}(\partial_x C_y - \partial_y C_x)\,. \tag{23}$$

Asymptotically (for $r \to \infty$), we take σ to be θ independent (this can be done because the geometric curvature vanishes), and

$$\oint C_\theta\,\mathrm{d}\theta = -\iint k\,\mathrm{d}(\text{area}) \equiv -\Phi_k\,. \tag{24}$$

We find that, for $r \to \infty$, $\sigma \to -(\Phi_k/2\pi)\ln r \equiv -n_k \ln r$. We already know that asymptotically, for $r \to \infty$, $\varphi \to (n_B + n_\phi)\ln r$, while for $r \to 0$, $\varphi \to n_\phi \ln r$.

First, let us assume that $\phi = 0$. In this case, $f(z)$ and $g(\bar{z})$ are polynomials. The solutions (21) to (20) have the following asymptotic form at infinity:

$$\psi_1 \to r^{-(n_B-(n_k/2))} f(z) \qquad \psi_2 \to r^{(n_B+n_k/2)} g(\bar{z})\,. \tag{25}$$

The square-integrability condition: $\iint \mathrm{e}^{2\sigma}|\psi_i|^2 r\,\mathrm{d}r\,\mathrm{d}\theta < \infty$ yields that ψ_1 is a polynomial of degree $n_1 < n_B + \frac{1}{2}n_k - 1$, while ψ_2 is a polynomial of degree $n_2 < -n_B + \frac{1}{2}n_k - 1$. Because the surface is homeomorphic to the plane, $\frac{1}{2}n_k < 1$. Hence, for a given setting, we can have either solutions of the form $(\psi_1, 0)^t$ or $(0, \psi_2)^t$, depending on n_B, but not both.

Assume $n_B > 0$. Then, all the square-integrable solutions of the vacuum equation are of the form $(\psi_1, 0)^t$. Because $f(z)$ is a polynomial of degree less than $n_B + \frac{1}{2}n_k - 1$, the space of functions $f(z)$ is of dimension $\lfloor n_B + \frac{1}{2}n_k 2\rfloor$ and this is the ground-state degeneracy of the Schrödinger–Pauli operator. (For $n_B < 0$, the ground-state degeneracy is $\lfloor -n_B + \frac{1}{2}n_k\rfloor$.)

For $\phi \neq 0$, and $n_B > 0$, we again have only solutions of the form $(\psi_1, 0)^t$. In this case, $f(z) = \sum_{n=n_{\min}}^{n_{\max}} a_n z^n$ with $n_{\min} > n_\phi - 1$, and $n_{\max} < n_B + \frac{1}{2}n_k + n_\phi - 1$, the number of such independent polynomials being $\lfloor n_B + \frac{1}{2}n_k + n_\phi + \rfloor - [n_\phi]$.

Again, it is not difficult to generalize this result to the case where there is a conic singularity of total curvature $2\pi n_{\phi_k}$ at the origin. In this case, $n_{\min} > n_\phi + \frac{1}{2}n_{\phi_k} - 1$, and $n_{\max} < n_B + \frac{1}{2}n_k + n_\phi + \frac{1}{2}n_{\phi_k} - 1$.

Hence, for a surface which is homeomorphic to the plane (except for one possible conical point at the origin, having curvature $2\pi n_{\phi_k}$, $0 \leqslant n_{\phi_k} < 1$), with compactly supported magnetic field and curvature, in the presence of a flux tube ϕ at the origin, the ground-state degeneracy of the Schrödinger–Pauli operator is

$$GSD = \lfloor |n_B| + \tfrac{1}{2}n_k + n_\phi + \tfrac{1}{2}n_{\phi_k}\rfloor - [n_\phi + \tfrac{1}{2}n_{\phi_k}]\,. \tag{26}$$

(We could expect this form, the only difference between this equation and the corresponding one for the plane being the replacement of B by the 'effective' field $B + k/2$ or $B - k/2$, depending only on the sign of n_B. See also [Pnu].)

Notice that the ground state is polarized (all the states belonging to it have the same spin direction).

The adiabatic charge transport picture is almost identical to the one we had for the plane, only the values of the flux where the degeneracy 'jumps' change. Hence, we again have a flow toward the origin when we increase the flux adiabatically.

5. Adiabatic transport of spinors on a 'generalized' cylinder

We calculate the transport on a *general* surface with cylindrical 'ends', using the Atiyah–Patodi–Singer [APS] index theorem for manifolds with a boundary. As in the previous sections, we do this by calculating the ground-state degeneracy as a function of a parameter, which represents an infinite flux tube that threads the cylinder without piercing it. To get an intuitive understanding of the results we start by calculating the ground-state degeneracy on the flat cylinder in the presence of a non-constant, compactly supported magnetic field.

5.1. The ground-state degeneracy of spinors on a cylinder

We calculate explicitly the L^2 ground-state degeneracy of a spinor on a cylinder. The calculations here are similar to these of Stone [Sto].

The Dirac operator on the plane is

$$\not{D}=\begin{pmatrix} 0 & (\partial_x - \mathrm{i}A_x) - \mathrm{i}(\partial_y - \mathrm{i}A_y) \\ (\partial_x - \mathrm{i}A_x) + \mathrm{i}(\partial_y - \mathrm{i}A_y) & 0 \end{pmatrix}. \tag{27}$$

We want to consider a *cylinder* of radius 1, so we impose periodic boundary conditions in the x-direction: $\psi(x+2\pi) = \psi(x)$. We choose a vector potential $A = A(x, y)\,\mathrm{d}x$. We choose, asymptotically (for $y \to \pm\infty$) $A = A\,\mathrm{d}x$ (this can always be done because the field vanishes at infinity).

Because of the periodicity in x, the wavefunctions are of the form

$$\psi(x, y) = \sum_n \varphi_n(y)\,\mathrm{e}^{\mathrm{i}nx} \qquad \varphi(y) = \begin{pmatrix} f_n(y) \\ g_n(y) \end{pmatrix} \tag{28}$$

where n denotes an arbitrary integer. The zero modes of the Dirac operator are, *asymptotically*, of one of the following two types.

(1)

$$\psi(x, y) = \mathrm{e}^{\mathrm{i}nx} \begin{pmatrix} f_n(y) \\ 0 \end{pmatrix} \qquad \frac{\mathrm{d} f_n(y)}{\mathrm{d}y} = (n - A) f_n(y). \tag{29}$$

(Notice that $f_n(y)\,\mathrm{e}^{\mathrm{i}nx}$ solves, asymptotically, the equation $K(f(y)\,\mathrm{e}^{\mathrm{i}nx}) = 0$. Also notice that for the regions where there exists a magnetic field, generically, unless B is cylindrically symmetric, $A = \sum_m A_m(y)\,\mathrm{e}^{\mathrm{i}mx}\,\mathrm{d}x$. For these regions, the zero modes involve more than one Fourier frequency, because the equations for the $f_n(y)$'s become coupled.) Such functions $\psi(x, y)$ are square integrable if and only if $n - A$ changes sign from negative to positive as y goes from $-\infty$ to $+\infty$. Choose the gauge: $A_{-\infty} = \phi_{\mathrm{in}}/2\pi \equiv n_{\mathrm{in}}$, $A_\infty = \phi_{\mathrm{out}}/2\pi \equiv n_{\mathrm{out}}$. The L^2 condition is equivalent to the demand

$$n_{\mathrm{in}} > n > n_{\mathrm{out}}. \tag{30}$$

Hence L^2 solutions of this form exist if and only if $\lfloor n_{\mathrm{in}} \rfloor \geqslant \lceil n_{\mathrm{out}} \rceil$, and the number of such solutions is

$$N_n = \lfloor n_{\mathrm{in}} \rfloor - \lceil n_{\mathrm{out}} \rceil + 1 \equiv \lfloor n_1 \rfloor + \lfloor n_2 \rfloor + 1 \tag{31}$$

(we defined $n_1 \equiv n_{\mathrm{in}}$, $n_2 \equiv -n_{\mathrm{out}}$, and $\lfloor x \rfloor$ is the largest integer strictly smaller than x).

(2)

$$\psi(x, y) = \mathrm{e}^{\mathrm{i}nx} \begin{pmatrix} 0 \\ g(y) \end{pmatrix} \qquad \frac{\mathrm{d} g_n(y)}{\mathrm{d}y} = -(n - A) g_n(y) \tag{32}$$

where $g(y)\,\mathrm{e}^{\mathrm{i}ny}$ solves the equation $K^\dagger(g(y)\,\mathrm{e}^{\mathrm{i}ny}) = 0$.

Hence the L^2 condition gives the demand

$$n_{\mathrm{out}} > n > n_{\mathrm{in}} \tag{33}$$

Here, L^2 solutions exist if and only if $\lfloor n_{\mathrm{out}} \rfloor > \lceil n_{\mathrm{in}} \rceil$, and their number is

$$N_n = \lfloor n_{\mathrm{out}} \rfloor - \lceil n_{\mathrm{in}} \rceil + 1 \equiv \lfloor -n_1 \rfloor + \lfloor -n_2 \rfloor + 1. \tag{34}$$

We see that for a given Dirac operator, we get only one 'type' of solutions (either $\mathrm{Ker}_{L^2}K$ or $\mathrm{Ker}_{L^2}K^\dagger$ being empty). So, without loss of generality, let us assume that $\lfloor n_{\mathrm{in}} \rfloor \geqslant \lceil n_{\mathrm{out}} \rceil$, and calculate the number of the zero modes of K. We already wrote this number as $N_n = \lfloor n_1 \rfloor + \lfloor n_2 \rfloor + 1$, where n_i denotes incoming fluxes. Now, we write N_n in a different form, by using the following two facts.

(1)

$$n_1 + n_2 = \frac{1}{2\pi} \iint B \equiv \frac{\phi_B}{\phi_0} \tag{35}$$

the number of magnetic flux quanta through the surface.

(2) The incoming and outgoing fluxes through the surface include both the 'real' flux tubes one puts, and the additional Berry phases, corresponding to the holonomy of the up (down) spinor: when the spinor is parallel transported around the cylinder, it acquires a phase of $\pm\pi$, depending on its direction (this is a Berry phase in *real* space, rather than in a parameter space (see also [Pnu])). Denoting by $n^{\mathrm{T}}_{\mathrm{in,out}}$ the 'true' incoming and outgoing fluxes, we find: $n^{\mathrm{T}}_{\mathrm{in,out}} = n_{\mathrm{in,out}} + \frac{1}{2}$.

If we now define a 'modified' fractional part of x:

$$\{x\}_{1/2} = \begin{cases} \{x\} & \text{if } 0 \leqslant \{x\} \leqslant \frac{1}{2} \\ \{x\} - 1 & \text{if } \frac{1}{2} < \{x\} < 1 \end{cases} \tag{36}$$

$(-\frac{1}{2} < \{x\}_{1/2} \leqslant \frac{1}{2})$, we can write the number of zero modes in the form:

$$N_n = \frac{1}{2\pi} \iint B - \sum_{j=1}^{2} \{n_j^{\mathrm{T}}\}_{1/2} \tag{37}$$

(remember: n_j^{T}, $j = 1, 2$, denote the 'true' *incoming* fluxes).

Similarly, for the second case we get minus this expression. This is all we need to calculate the charge transport.

5.2. *Adiabatic transport on the cylinder*

To calculate the Hall conductance we 'put' a flux tube through the cylinder, and follow the flow of states as we adiabatically increase the flux from zero to one flux quantum ϕ_0.

The changing flux tube generates an infinitesimal electromotive force around the cylinder (in the $\hat{x}$-direction). Therefore, the 'Hall current' should flow in the $\hat{y}$-direction. Hence, the Hall conductance equals the net number of states transferred from $y = -\infty$ to $y = +\infty$ during the adiabatic increasing of a flux, ϕ, threading the cylinder, from zero to one flux quantum, ϕ_0.

If we introduce the flux through the boundary conditions, the effect of adding it is

$$n_1 \to n_1 + \frac{\phi}{\phi_0} \qquad n_2 \to n_2 - \frac{\phi}{\phi_0}. \tag{38}$$

One can easily see that, generically, there are two 'jumps' in the degeneracy of the ground state during the process, when either $n_1 + (\phi/\phi_0)$ or $n_2 - (\phi/\phi_0)$ cross an integer. In the first case the degeneracy *decreases* by one, while in the second case it *increases*, again by a single state. At the end of the process the degeneracy returns to its original value, as it should. But during the process the states have flowed: when $n_1 + (\phi/\phi_0) = \text{integer} + \varepsilon$ (ε denotes a positive infinitesimal number), a state which was not in L^2, because of exponential divergence at $y \to -\infty$, becomes an 'honest' L^2 state, while when $n_2 - (\phi\phi_0) = \text{integer} + \varepsilon$

an L^2 state becomes a non-normalizable state that diverges exponentially as $y \to +\infty$. From this we deduce that the net effect of the process is transferring one state from $-\infty$ to $+\infty$ during the process. We say that the 'Hall conductance' for the ground state of the Schrödinger–Pauli operator on the cylinder is exactly 1.

5.3. Hall conductance and the APS index theorem

Let us examine again the case of the infinitely long cylinder from a different point of view: because we assume that there is a magnetic field only on a finite part of it, it follows that one can 'cut' out of it a *finite* cylinder, such that this finite cylinder includes all the area with non-vanishing magnetic field. It turns out that the number of zero-modes which obey the 'APS boundary conditions' (to be defined) on this finite cylinder (almost) equals the number of L^2 zero-modes on the (original) infinite cylinder. This is true for all surfaces with 'cylindrical' ends. For such surfaces, we can find $\mathrm{Ind}_{L^2} \not{D}$, by calculating the Dirac index $\mathrm{Ind}\,\not{D}$ *for manifolds with boundaries, using the Atiyah–Patodi–Singer scheme* [APS], where the manifold with the boundary is a compact sub-manifold of the original surface, containing all the area on which either B or k do not vanish.

Near the boundary, the Dirac operator can be written in the form

$$\not{D} = \begin{pmatrix} 0 & (\partial/\partial u) + b \\ -(\partial/\partial u) + b & 0 \end{pmatrix} \tag{39}$$

where u is the coordinate perpendicular to the boundary, and b is the differential operator—'the boundary operator'. In our case, we choose $b = -\mathrm{i}\partial_\alpha - \phi_{(\mathrm{in}}/\phi_0)$, where α parametrizes the boundary.

Here we only quote the APS result (for a general compact surface, with boundaries)

$$\mathrm{Index}\,\not{D} = \frac{1}{2\pi}\iint B - \frac{1}{2}\sum(\eta(0) + h) \tag{40}$$

(details can be found in [APS] physically oriented references are [FOW, NS]).

The 'correction' term to the index, compared with the index on manifolds without boundaries, is associated with the boundary operator. $\eta(0)$ is defined by $\eta(s) = \sum_{\lambda_n \neq 0} \mathrm{sign}(\lambda_n)|\lambda_n|^{-s}$, where λ_n are the eigenvalues of b, and h is the number of zero modes of b. $\eta(0)$ is called the *spectral asymmetry* of the boundary operator. Take, for example, $b = -\mathrm{i}\partial_\theta - (\phi_{\mathrm{in}}/\phi_0)$. Then

$$\eta(0) = \begin{cases} 2\{(\phi_{\mathrm{in}}/\phi_0)\} - 1 & \text{if } \{(\phi_{\mathrm{in}}/\phi_0)\} \neq 0 \\ 0 & \text{if } \{(\phi_{\mathrm{in}}/\phi_0)\} = 0 \end{cases} \tag{41}$$

and

$$h = \begin{cases} 1 & \text{if } \{(\phi_{\mathrm{in}}/\phi_0)\} = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{42}$$

Hence

$$\begin{aligned} \tfrac{1}{2}(\eta(0) + h) &= \begin{cases} \{(\phi_{\mathrm{in}}/\phi_0)\} - \frac{1}{2} & \text{if } \{(\phi_{\mathrm{in}}/\phi_0)\} \neq 0 \\ +\frac{1}{2} & \text{if } \{(\phi_{\mathrm{in}}/\phi_0)\} = 0 \end{cases} \\ &= \begin{cases} \{(\phi^{\mathrm{T}}_{\mathrm{in}}/\phi_0) + \frac{1}{2}\} - \frac{1}{2} & \text{if } \{(\phi_{\mathrm{in}}/\phi_0)\} \neq 0 \\ \{(\phi^{\mathrm{T}}_{\mathrm{in}}/\phi_0)\} & \text{if } \{(\phi_{\mathrm{in}}/\phi_0)\} = 0 \end{cases} \equiv \{(\phi^{\mathrm{T}}_{\mathrm{in}}/\phi_0)\}'_{1/2} \end{aligned} \tag{43}$$

where $-\frac{1}{2} \leqslant \{x\}'_{1/2} < \frac{1}{2}$ denotes another 'modified fractional part' of x (compare with (36)). Hence the index of the Dirac operator on the manifold is

$$\mathrm{Index}\,\not{D} = \frac{1}{2\pi}\iint B - \sum_j \{n^{\mathrm{T}}_j\}'_{1/2} \tag{44}$$

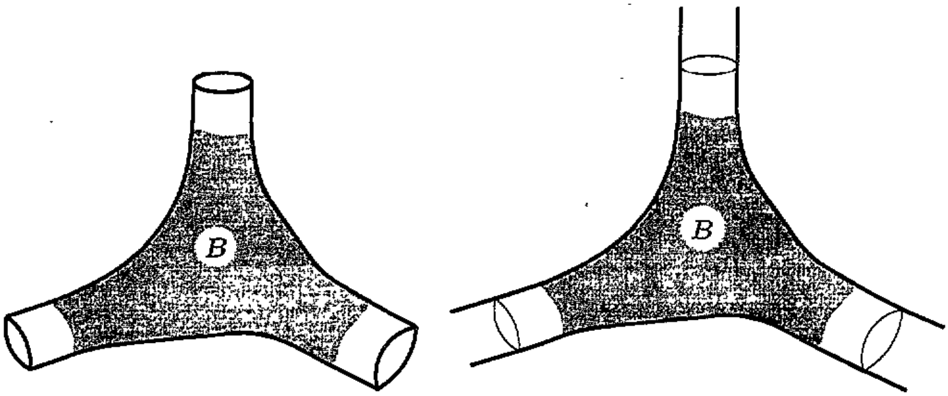

Figure 2. (*a*) A compact surface with boundaries and cylindrical 'ends', with an applied magnetic field. (*b*) The 'corresponding' infinite surface.

where $n_j^{\mathrm{T}} \equiv \phi_j^{\mathrm{T}}(\mathrm{in})/\phi_0$ denotes the number of incoming flux quanta through the boundary j.

However, it is known (see [APS]) that

$$\mathrm{Index}\, \not{D} = \mathrm{Index}_{L^2}\, \not{D} - h \tag{45}$$

(where Index $\not{D}$ is the index of the Dirac operator on the compact surface, while $\mathrm{Index}_{L^2}\, \not{D}$ is the L^2 index of the Dirac operator on a non-compact surface, which can be obtained from the compact one by gluing to it semi-infinite cylinders. In our example the compact surface is the finite cylinder, and the non-compact one is the infinite cylinder). Hence

$$\mathrm{Index}_{L^2}\, \not{D} = \frac{1}{2\pi} \iint B - \sum_j \{n_j^{\mathrm{T}}\}_{1/2}\,. \tag{46}$$

This result coincides with the one we got for the cylinder by a direct computation (see (37)). But the result here is applicable to a much wider class of surfaces. An example is shown in figure 2, where we draw both the compact surface with boundaries to which we apply the APS index theorem, and the non-compact extension of it.

Calculating the charge transport is now simple: as in the case of the cylinder, we vary adiabatically a flux tube that goes through the surface between two boundaries. The index 'jumps' twice: once we loose a state, and once we gain one. Hence, we would like to conclude that there was one *net* state flowing between the two 'ends'. But this is not so simple because one has to prove that we did not gain or lose *pairs* of states, having opposite spin direction, during the process, because the index is insensitive to such a situation. For this not to happen we can give sufficient conditions (mainly, the magnetic field plus half the Gaussian curvature being either positive or negative *everywhere* (see [Pnu])), but we know from the example of the infinite cylinder that this condition is not necessary (we found that one of the kernels is empty, regardless of any local configurations of the magnetic field, depending only on the total magnetic flux through the surface).

Finally, we mention that we can relax the assumption of the 'cylindrical ends'. In the case of conical ends we also get two 'jumps' of the index during the adiabatic process, only their locations as a function of the flux change, according to the curvature of the boundary. A condition that we cannot relax is the demand that near the boundary there is an 'epsilon neighbourhood' with no magnetic field or geometrical curvature.

6. Summary

In this paper, we considered charge transport due to an adiabatic variation of a magnetic flux tube on various non-compact surfaces. We demonstrated how to use knowledge on the *ground-state degeneracy* in order to get information on the *charge transport*. For this we also need to know what happens at the critical values of the flux, the values where we 'gain' or 'loose' an L^2 state, in order to know to where (or from where) a state has been transported.

Because we deal with spinors the ground-state degeneracy is related to the Atiyah–Patodi–Singer index. We connected the charge transport on non-compact surfaces with the jumps in the value of an eta invariant for the boundary of a corresponding *compact* surface: the values of a flux tube for which the ground state of the non-compact surface jumps are exactly the values for which (at least one) eta invariant of the compact surface jumps.

Interpreting the adiabatic changing of the flux as a source of an 'infinitesimal' electromotive force enables us to deduce from the transport the appropriate (averaged) Hall conductance.

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